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Hamiltonian Structure of Gravitational Field Theory

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Hamiltonian generalizations of Einstein's theory of gravitation introducing a laminar structure of spacetime are discussed. The concepts of general relativity and of quasi-inertial coordinate systems are extended beyond their traditional scope. Not only the metric, but also the coordinate system, if quantized, undergoes quantum fluctuations.

1. INTRODUCTION

The problem of reconciling general relativity (GR) with quantum theory is still open. The main difficulties may be seen in the fact that Einstein's general theory of relativity is not canonical in the usual sense of this word: the metric tensor components $g_{0\mu}$ ($\mu=0, 1, 2, 3$) are not canonical variables, since the Lagrangian is not quadratic in the time derivatives $\dot{g}_{0\mu}$, in contradistinction to g_{kl} ($k, l=1, 2, 3$). The Lagrangian does not contain $\dot{g}_{00}$ at all and is only linear in the three quantities $\dot{g}_{0k}$. Hence, the equations defining the canonically conjugate momenta

$$\pi^{0\mu}=\frac{\partial \mathscr{L}}{\partial \dot{g}_{0\mu}} \tag{1}$$

are not soluble with respect to $\dot{g}_{0\mu}$. Consequently, $g_{0\mu}$ are to be regarded as variables of constraints, which presents serious difficulties with quantization.

Serious difficulties connected with defining energy-momentum and the Hamiltonian for the pure gravitational field arise as well. The Einstein quantities τ^{μ}_{ν} possibly representing the energy-momentum-stress tensor of the free gravitational field do not transform like genuine tensor components; therefore, τ^{μ}_{ν} was called a "pseudotensor." Moreover, the energy density τ^{0}_{0} may be made to vanish at an arbitrary point of spacetime by a mere

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coordinate transformation, so that energy in GR is to be regarded as nonlocalizable. According to the arbitrariness of the choice of the gauge, i.e., of the coordinate system, one gets different and inequivalent expressions for the generator of infinitesimal translations in spacetime which in other field theories play the role of energy-momentum. The above-mentioned properties are closely related with the nonexistence of privileged coordinate systems playing a similar role to inertial ones in special relativity.

If one tries to perform a canonical formulation in an obviously and explicitly gauge-invariant form, i.e., independently of a choice of the coordinate system, one gets a startling result: The Hamiltonian of a free gravitational field is found to be zero, $H=0$. From this it follows that the dynamics of explicitly covariant gravity must be frozen! This fact is explicable in the following way: inasmuch as in GR all one-parameter families of spacelike hypersurfaces $t=\text{const}$ are on equal footing, no privileged lapse of time algorithm exists and Nature "cannot decide" for any specific choice of an infinitesimal translation in the timelike direction.

2. A CANONICAL EXTENSION

The way out of these difficulties that we propose is radical: we outfit spacetime with a kind of laminar structure and relax Einstein's principle of general covariance (GC). This may be done in the following way: Let us add to the usual Lagrangian $\mathscr{L}$ of GR an additional term

$$\mathscr{L} \to \mathscr{L} + \tilde{\mathscr{L}} \qquad \text{where} \quad \tilde{\mathscr{L}} = \frac{\varepsilon}{2} U_\mu g^{\mu\nu} U_\nu \tag{2}$$

with U_ν being linear functions of the time derivatives $\dot{g}_{0\mu}$. Interesting examples are

$$U_\nu = \partial^\mu (\sqrt{-g}\, g_{\mu\nu}) \tag{3'}$$

and

$$U_\nu = \partial_\nu \sqrt{-g} \tag{3''}$$

where g is the determinant of the metric tensor components.

Obviously, the introduction of such additional terms into the Lagrangian spoils its covariance and yields modified field equations

$$G^{\mu\nu} + \tilde{G}^{\mu\nu} = \kappa T^{\mu\nu} \tag{4}$$

which do not satisfy Bianchi identities and, consequently, determine not only the geometry of spacetime, but also a privileged class of coordinate systems, or in the case (3″) at least one privileged coordinate, viz. the time coordinate.

The loss of GC is, however, richly recompensated by the advantage of getting a genuine Hamiltonian formalism admitting a vast class of canonical transformations. A general covariance under coordinate transformations has been lost, but this loss has been compensated by another general covariance: under canonical transformations!

It should be noticed that by specializing the initial conditions so that the terms U_ν become equal to zero

$$U_\nu = 0 \tag{5}$$

one has introduced a further specialization of coordinates. In the case (3′) or (3″) this means either

$$\partial^\mu(\sqrt{-g}\, g_{\mu\nu}) = 0 \qquad \text{or} \qquad \partial_\nu \sqrt{-g} = 0 \tag{3a}$$

the former denoting de Donder–Fock coordinates, while the latter is satisfied by putting $g = \text{const}$, which means invariance under unimodular coordinate transformations. This causes the additional terms in the field equations to vanish, so that their usual form is regained,

$$G^{\mu\nu} = \kappa T^{\mu\nu} \qquad \text{if} \quad U_\lambda = 0 \tag{6}$$

which follows from the fact that $\tilde{\mathscr{L}}$ is bilinear and $\tilde{G}^\mu_\nu$ linear in U_λ. From this it is seen that the generalized formalism contains all the Einstein solutions, i.e., *all Einsteinnian spacetimes*, although they appear described in specialized coordinates. Hence, the postulate of correspondence with Einstein's original theory is still satisfied. Being truly canonical and Hamiltonian, such a formalism is quantizable, at least formally, disregarding the convergence problems.[2]

3. THE PROBLEMS OF GENERAL RELATIVITY

The next problems are connected with the circumstance that there exists an infinite variety of possible choices of the term U_ν, all of them yielding Lagrangians quadratic in the time derivatives of all metric tensor components, and there appears the question of their equivalence or inequivalence. If they are not equivalent, then there arises the problem of selecting a proper form of U_ν. In quantum GR there appears the fundamental problem of dividing physical reality into two parts: the object of measurement and the apparatus together with well-defined measurement conditions (preparation of the act of measurement). A system of reference together with a comoving coordinate system belongs to the latter. The choice of a particular U_ν must be dictated by a particular division of physical reality into subjective and

[2]This problem may be solved with the help of the concept of supersymmetry.

objective aspects of observations whereby two different choices may be equally well acceptable although they do not need to be unitarily equivalent. *A priori* there exist two alternatives: one is to regard all possible forms of U_ν on an equal footing, even if they are inequivalent, whereby the existence of a variety of different formalisms obtained in this way may be just interpreted as reflecting a generally relativistic character of the theory: even the Lagrangian is not to be assumed to be an absolute element of the formalism (i.e., does not need to be a scalar or differ from a scalar by a divergence) and does not need to be *a priori* fixed in a unique way, but it may be adjustable relative to the assumed partition of physical reality into object and apparatus of measurement. *This means a radical modification of the concept of relativity.*

The other possibility is to admit that in principle we are unable to construct an arbitrary apparatus of measurement and to secure arbitrary means of observation, but have to make a special choice of U_ν and, consequently, of the Lagrangian. Thus, our limited means of observation might lead to specialized U_ν and, consequently, specialized frames of reference and coordinate systems fixed to the bodies forming the reference frame. In such cases a privileged role of some types of frames of reference and of comoving coordinates should be also distinguished on some aesthetic basis: either assuming (3′) and its vanishing, which denotes the introduction of de Donder–Fock coordinates constituting a natural generalization of Lorentz conditions so much distinguished in electrodynamics, or vanishing (3′), which means that the generalized theory is invariant under the group of unimodular coordinate transformations $x^\mu \to \bar{x}^\mu$ satisfying the condition

$$\det \frac{\partial \bar{x}^\mu}{\partial x^\nu} = 1 \tag{7}$$

The unimodular group is the most natural and most general subgroup of Einstein's group of general transformations enabling us to treat tensor densities on an equal footing with tensors. General covariance in its traditional sense has been destroyed, but it is intelligible in a quantum theory of gravity, where the choice of coordinates not only means attaching arbitrary names to the points of spacetime, but also a partition of physical reality into the object and the means of observation, into substance and form.

Certainly, it is difficult to accept a formalism which rejects the traditional dogma of general covariance. However, since this new formalism describes equally well all crucial effects—red shift, deflection of light, and perihelion motion of Mercury—such objections cannot be regarded as decisive and conclusive and such reservations can only be regarded as dogmatic in character.

Moreover, a general covariance may be still introduced, so to say "by the back door," if one assumes that the solutions $g_{\mu\nu}$ of the field equations form a tensor under general coordinate transformations and transforms these solutions (but not the Lagrangian!) in an arbitrary way. This is already sufficient to satisfy the relativistic requirement, namely that physical phenomena should be independent of the mode of their description, or of the names attached to the points of spacetime.

4. SUPPLEMENTARY CONDITIONS

Equations (4) derivable directly from the canonically extended formalism are poorer in their form than the traditional equations of Einstein since they have been expressed in specialized coordinates determined also from these equations. By specializing these coordinates still more by means of the supplementary conditions (5), these equations regain their traditional form (6) at a cost of a still greater restriction of the class of privileged coordinate systems. The conditions (5) may be regarded as constraints, but of a weak form, since they do not modify the original solutions, but merely select their subclass.

The conditions (5) are easy to introduce within the framework of the classical formalism, but satisfying them in the quantized version of the theory is problematic. First of all, they cannot be satisfied as conditions upon the operators themselves, but only as conditions imposed upon the state vectors. Then we run a risk of getting inconsistencies, since it cannot be *a priori* guaranteed that for any type of the operator U_ν their commutators will not produce again and again new conditions upon the state vector. Therefore, we prefer not to introduce any supplementary conditions at the whole time interval, but to introduce only suitable initial (and/or final) conditions.

5. INERTIAL COORDINATES IN CURVED SPACETIMES

The main interpretational difficulties of GR are the absence of local conservation laws for energy-momentum and the nonexistence of inertial coordinate systems. The energy-momentum tensor of material sources T^μ_ν does not satisfy a continuity equation, while the energy-momentum of the free gravitational field is not a genuine tensor at all.

The continuity equation for T^μ_ν alone could be secured and τ^μ_ν could be dispensed with if the following four conditions are satisfied:

$$\Gamma^\lambda_{\mu\nu} T^{\mu\nu} = 0 \tag{8'}$$

where $\Gamma^\lambda_{\mu\nu}$ are Christoffel symbols. Indeed, the above formulas are equivalent to the ordinary continuity equations for the tensor density

$$\partial_\mu(\sqrt{-g}\, T^{\mu\nu}) = 0 \tag{8''}$$

If the formulas (8) hold, then the energy-momentum of the sources alone is locally conserved. Consequently, no exchange of energy-momentum between the sources and the free gravitational field (waves) would occur. Thus, the above formulas may be satisfied only in the cases of gravitationally stationary fields.

Definition. If by a special choice of the coordinate system the conditions (8) may be satisfied, then the field is gravitationally stationary and the respective system of coordinates will be called *inertial.*

Indeed, the name "inertial" is justified since it is only by transition to any other coordinate system that some pseudodynamical effects could appear imitating the emission, absorption, and scattering of gravitational waves, but interpretable as nothing else but coordinate effects.

Thus, the concept of inertial coordinate systems makes sense even in the presence of gravitation, but it is limited to gravitationally stationary fields. Nevertheless, it may be shown that this idea is useful even in the general case of arbitrary, nonstationary, or turbulent fields, since we may introduce some coordinate systems as similar as possible to inertial ones and call them "quasiinertial." Their introduction will enable us to distinguish mere coordinate effects from truly dynamical ones even in the cases of strongly nonstationary field configurations.

6. FUSION OF CANONICAL EXTENSIONS WITH THE CONCEPT OF INERTIA

In Section 2 a generally canonical (instead of generally covariant) formalism was formulated by supplementing the Lagrangian by an additional term (2) involving quadratically the terms U_λ involving $\dot{g}_{0\mu}$ linearly and such that their vanishing means a special choice of coordinates. This formalism may be combined with the idea of inertial systems described in Section 5 in the following way: Inasmuch as equations (4) are second-order equations for all metric tensor components $g_{\mu\nu}$, we may dispose freely of the initial values of $g_{0\mu}$ (or at least of g_{00}) as well as of their first-order time derivatives, whereby the coordinate system will become uniquely determined except for suitable boundary conditions at spacelike infinity. Instead of initial conditions for $g_{0\mu}$ and their time derivatives, we may introduce initial and final conditions at the two time instants t_i and t_f. We shall choose such mixed initial–final conditions in such a way that the coordinate system will become instantaneously inertial at these time instants

$$\Gamma^{\lambda}_{\mu\nu}T^{\mu\nu}=0 \qquad \text{at} \quad t_i \text{ and } t_f \tag{9}$$

A coordinate system following from equations (4) supplemented by the conditions (9) deserves to be called "quasi-inertial." In classical theory it exists always, even in the case of nonstationary, turbulent fields. Nonetheless, neither emission nor absorption nor scattering of free gravitational waves is observable at the very instants of initial and final measurements.

In quantum theory the situation is not so simple. The conditions (9) must be regarded as conditions upon the state vector and one has a problem of their compatibility. But it is also sufficient to assume only one condition at the initial time instant

$$\Gamma^{0}_{\mu\nu} T^{\mu\nu} = 0 \qquad \text{at} \quad t_i \tag{10}$$

in order to secure a quasi-inertial situation at the initial instant of state preparation and to avoid the appearance of gravitons *in statu nascendi.* The coordinate system satisfying such initial condition is not fully determined, which accounts for quantum fluctuations not only of the metric, but also of the system of reference.